

Price drift as an outcome of differences in higher order beliefs

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Abstract

Motivated by the insight of Keynes (1936) on the importance of higher order beliefs in financial markets, we examine the role of such beliefs in generating drift in asset prices. We show that in a model in which agents have heterogeneous priors and are uncertain about the beliefs of others, differences in higher order beliefs may lead to price drift. Such drift does not arise in the classical difference of opinion paradigm, in which others' beliefs are common knowledge. Our results stand in contrast to Allen, Morris and Shin (2006), as we argue that in rational expectation equilibrium, heterogenous beliefs do not lead to the failure of the law of iterated expectations and price drift

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1 Introduction

Post earning price drift and momentum, first documented by Ball and Brown (1968), Jegadeesh (1990), Lehmann (1990) and Jegadeesh and Titman (1993), are some of the most intriguing and empirically robust findings about stock price dynamics. These phenomena imply that conditional on past performance, future stock returns continue to drift in the same direction. Recent empirical evidence suggests there exists a link between price drift and heterogeneity of beliefs. Specifically, price drift arises when there is greater disagreement among investors (e.g. Zhang (2006) and Verardo (2006)).

A common, and somewhat casual, explanation for this relationship is based on the intuition that in the presence of noise, prices are slow to aggregate information and, as a result, prices drift slowly towards the fundamental value. A more involved explanation is based on the notion that in a dynamic rational expectations equilibrium (REE hereafter) model, investors may need to forecast the forecasts of others as in Keynes' (1936) "beauty contests." Again, due to noise in investors' signals, prices aggregate information slowly. In a classical paper, Townsend (1983) argues that in a dynamic REE model with production, agents form beliefs about the beliefs of others and this leads to serial correlation in unconditional forecast errors. More recently, Allen, Morris and Shin (2006) (AMS hereafter) make a similar argument in a REE model of financial markets. They argue that "failure of the law of iterated expectation" leads to a violation of the Martingale property that results in a drift in prices. They relate their finding to Keynes' (1936) "beauty contest" metaphor, which implies that investors may buy a stock not only because they consider it to be attractive, but also because they believe other investors do.

The goal of this paper is to take a closer look at the role of this slow aggregation of heterogeneous beliefs channel in generating price drift. Our results challenge the views mentioned above. While heterogeneous beliefs in a multi-period model indeed make higher order beliefs relevant, this is not sufficient to generate a price drift. in a REE, as agents observe the price and use it to correct any bias such as the "failure of the law of iterated expectation" that AMS refer to. To obtain a price drift one also needs to

assume that investors “agree to disagree,” or have “differences of opinions” (DOO hereafter). The contribution of our paper is two-fold:

1) We examine what conditions give rise to the phenomenon that AMS had in mind. We show that it arises in DOO models, as in these models agents may not fully condition on the price. While the ‘standard’ DOO model, where opinions are common knowledge, delivers such a result in a static model, this is not the case in a dynamic setting. In a dynamic model agents choose to extract information from prices. As a result, with either common knowledge or disagreement only about first order beliefs there is no price drift. For price drift to arise one needs to assume higher order difference of opinions. It is important to note that in the economy we examine despite the disagreements investors pay close attention to prices. They use prices to infer what are the opinions of other investors

2) In contrast to AMS, we argue that in a REE (at least in the model they consider) there is no price drift. As mentioned above, this is due to the fact that in REE agents observe the price and hence correct for the bias suggested by AMS. The different conclusion is in part an artifact of the way price drift is defined. While AMS consider an ex-post notion that depends on realized price paths, we argue that one should use an ex-ante measure where one conditions only on past prices. We discuss this important distinction in more detail in section 4.

Our paper also contributes to the ongoing debate between REE and DOO models. The fact that investors hold heterogeneous beliefs has long been recognized as a key factor in financial markets. The two major paradigms for modeling this heterogeneity are REE and DOO. Both approaches share the view that investors have different valuations, and prices aggregate the different views during the trading process; they differ in whether agents can agree to disagree. Agents that agree to disagree hold different views if when their views become common knowledge they still disagree. In other words, it is commonly known that one investor believes that a given stock is attractive while another investor does not. Clearly in most cases these views are not commonly known but still this assumption has a strong implication for how investors interact, specifically

it effects what information they extract from the price. The REE paradigm rules out such disagreements and agents hold different views only because of private information. The DOO paradigm allows for such disagreements. Some categorize DOO ‘behavioral’ or ‘irrational’ but this need not be the case. For example, Aumann (1979) notes in his classic work that rational people are likely to agree to disagree. Still, in recent years with the rising interest in behavioral economics we see a renewed interest in models based on differences of opinions (e.g. Scheinkman and Xiong 2003). The reason for this interest is that allowing for disagreements has an intuitive appeal, and also seems to be necessary for explaining trade.

Models that are based on DOO typically assume that the different views are common knowledge (e.g. Harrison and Kreps (1978), Harris and Raviv (1989), and Kandel and Pearson (1995)). Although primarily made for tractability, this strong assumption is somewhat unnatural. Given the uncertainty regarding fundamentals, it is not clear how investors are certain about other agents’ opinions. We show that this common knowledge assumption eliminates the link between heterogeneous beliefs and price drift. We then relax the common knowledge assumption and instead assume that there is also uncertainty about the average opinion. While investors ignore the opinions of others in estimating the value of the asset, they do understand that these views may influence intermediate prices (see also Cao and Ou-Yang (2005)), and as such have a direct impact on their ability to profit from speculating on future prices.. Investors learn from prices like in a REE, but only to update their beliefs about the average opinion. They trade based on their views about the fundamentals and their beliefs about what other investors think about these fundamentals. We show that in such a model, there is a drift in prices when investors agree to disagree about the average valuation. Hence, price drift is a result of differences in higher order beliefs.

The rest of the paper is organized as follows. In Section 2, we introduce the basic notation and as a first step compare the classical REE model to a DOO model, in a two-date model. In particular, a DOO setting leads naturally to price drift as agents do not condition on prices when deciding how much to invest in the risky asset, but the REE does not. While the static model demonstrates why DOO models are suitable for

generating price drift as a result of aggregation of heterogeneous beliefs, it is not suitable to examine implications of higher order beliefs. For higher order beliefs to come into play, one needs to assume that investors live for more than one period.

Section 3 presents the main analysis of the paper. We study a dynamic DOO model with long lived investors who have differences of opinion about the value of the risky asset. In addition, they are also uncertain about the views of other investors. This gives rise to higher order beliefs, and also creates a REE-type, “learning from prices” feature. While investors may hold strong views about the fundamental value of the asset, they realize that others influence intermediate prices. Hence, each investor infers what others think from the current price to speculate on intermediate prices. Our main result in this section (which is also the main conclusion of the paper) is that higher order difference of opinions leads to drift in asset prices. However, if investors do not have differences of opinion about average beliefs, then there is no drift.

Section 4 discusses related literature, starting with a discussion of why our results differ from those in AMS. As we discuss there, the two papers have different notions of price drift. We use an ex-ante definition that conditions only on information available to the agents within the model at the time they make their investment decisions; requiring that higher prices today on average be followed by higher price changes in the future. AMS, on the other hand, implicitly condition on the assets terminal value, showing prices drift towards that realized value. Section 5 concludes.

2 Static (myopic) Setup

We begin our analysis by considering a two-date static model based on Grossman (1976). We argue that heterogeneity of beliefs does not induce price drift in a REE; such drift is present when there are differences of opinion. The model in this section also serves as a benchmark for the dynamic model that we examine in the following section.

We describe the model in general terms since we will use the same basic structure to consider different static cases in this section, and extend it to multiple periods in the

next section. We assume that at $t = 1$ a continuum of agents, indexed by i , trade two assets: a risk-free asset, whose net return is normalized to zero; and a risky asset that pays V at $t = 2$. Agents share a CARA utility function with a risk aversion coefficient γ . As a result, agent i , who faces a current price P_1 for the risky asset, chooses his position, x_1^i , by solving:

$$x_1^i \in \arg \max_x E[-\exp(\gamma x(V - P_1)) | I^i] \quad (1)$$

where I^i denotes his information set (including the current price).

Agents share a common prior on the distribution of the asset payoff, given by

$$V \sim N(0, 1/\rho_{V,0}).$$

In addition to the common prior, agent i also receives a conditionally independent signal

$$S^i = V + \varepsilon^i \quad (2)$$

where $\varepsilon^i \sim N(0, 1/\rho_\varepsilon)$. Based on his information set I^i agent i forms his posterior beliefs:

$$V | I^i \sim N(E^i[V], 1/\rho_V).$$

Note that the symmetry in the agents' information sets implies that the posterior precision is the same across all agents. Our differences of opinion (DOO) version of the model differs from the REE version in that agents agree to disagree. Agent i disregards other signals and so sets his posterior to $V | S^i$. This assumption is common to models based on difference of opinions (see for example Harrison and Kreps (1978)) and is the key departure from a classic REE setup. Each agent believes that no other agent holds information of any additional value to his private information. In contrast, in an REE framework (see for example Hellwig (1980)), agents believe that $V = \int_i S^i di$ and so they place a big weight on other agents' information when determining the asset's value. One can also view our model as a model with heterogeneous priors; the signals represent these different priors.¹

¹ One can also examine model that is not as extreme in which agents put a positive weight on other signals. This does not change the qualitative conclusions we draw.

We focus on price drift or positive autocorrelation in prices. We take the final price to equal the realization of the end-of- period payoff and the prior price to equal the prior mean of that payoff, i.e. we set $P_0 = 0$ and $P_2 = V$.² We shall say that prices exhibit *price drift* if $E_1[P_2 - P_1 | P_1] = E_1[P_2 - P_1 | P_1 - P_0]$ is increasing in $P_1 - P_0$. In section 3.1 we extend this definition to the multi-period case.

2.1 Analysis

The starting point is the known fact that under the joint assumptions of normally distributed payoffs and CARA utility we have that:

$$x_0^i(P_1) = \frac{\rho_V}{\gamma} (E^i[V] - P_1) \quad (3)$$

Note that (3) holds in both REE and DOO; the difference between the two versions is in the way agents form their conditional beliefs, $E^i[V]$ and ρ_V .

If there is no aggregate noise in the economy then assuming net supply is zero implies that the market clearing condition is given by $\int_i x_1^i(P_1) di = 0$ and we have:

$$P_1 = \int_i E^i[V] \equiv \bar{E}[V] \quad (4)$$

Recall the classic REE result by Grossman (1976) who showed that $P_1 = V$ and as a result there is no drift. This is a simple demonstration that in REE the intuition we described in the introduction is not valid. This is because the current price is part of the public information that agents use to update their beliefs.

In contrast, when agents exhibit a difference of opinions they put more weight on their signal and less on the information held by others, which is reflected in the price. In a static model agents will choose to ignore the information in prices and set:

$$E^i[V] = \frac{\rho_\varepsilon}{\rho_\varepsilon + \rho_{V,0}} S^i$$

which implies that the equilibrium price is given by:

$$P_1 = \frac{\rho_\varepsilon}{\rho_\varepsilon + \rho_{V,0}} \int_i S^i = \frac{\rho_\varepsilon}{\rho_\varepsilon + \rho_{V,0}} V$$

² Note that all agents are ex-ante identical. Thus, $P_0 = 0$ is the price that would prevail in the market if agents were allowed to trade at that point.

Since $\frac{\rho_{V,0}}{\rho_\varepsilon} > 0$, we have price drift:

$$P_2 - P_1 = \frac{\rho_\varepsilon + \rho_{V,0}}{\rho_\varepsilon} P_1 - P_1 = \frac{\rho_{V,0}}{\rho_\varepsilon} (P_1 - P_0).$$

A standard assumption in the literature is the existence of noise traders. The addition of noise makes the REE version of the model equivalent to the noisy REE model in Hellwig (1980). In both the REE and DOO versions, when there is aggregate noise the market clearing condition market is given by $\int_i x_1^i(P_1) di = Z_1$ where Z_1 is the noise at time $t = 1$. In this case the price is given by:

$$P_1 = \bar{E}[V] - \frac{\gamma}{\rho_V} Z_1 \quad (5)$$

where $\bar{E}[V]$ and ρ_V depend on whether the equilibrium is REE or DOO.

One may suspect that the introduction of noise would change our conclusions in the REE setup. This is based on the idea that as a result of noise, the information is slowly incorporated into prices and this leads to drift or momentum. However, we argue that this again is not true; moreover, noise induces negative autocorrelation in prices for the REE setup. In the DOO version, since noise induces negative correlation in price, we have a drift in a static model only under certain conditions. Specifically, we argue that:

Proposition 1 (i) In REE prices exhibit reversals, that is $E[P_2 - P_1 | P_1] = \alpha P_1$ for some $\alpha < 0$, (ii) In DOO there is price drift if and only if $\rho_Z \rho_\varepsilon - \gamma^2 > 1$, where ρ_Z is the precision of the noisy supply

3 A dynamic setup

The analysis in the previous section is limited, as it considers a single period model. We have showed in this very simple setup that while REE does not yield a price drift or momentum, DOO may. In this section, we examine the effect of difference of opinions in a multi-period model. A key difference is that in a multi-period model, agents care about the opinions of other agents, even when they exhibit DOO. This is because the views of other agents affect intermediate prices, which agents can speculate on. As a result the equilibrium is affected by “beauty contest” considerations and higher order

beliefs play a key role. Our main result is that a simple DOO structure does not yield a drift in multi period model, except for the last period. A necessary and sufficient condition in the dynamic model is the presence of higher order differences of opinions. That is, agents not only agree to disagree about the asset’s value but also about what the average views are.

3.1 Model

The model we examine is based on the one introduced previously; we simply add one more date at which agents can trade. Specifically, the value of the risky asset is realized at $t = 3$ and agents trade at $t = 1, 2$. While the addition of one trading round at $t = 2$ is the key change from our previous model we should make some additional comments.

1) Unlike standard DOO models, we assume that agents do not observe the opinions of others. In a DOO model, agents care about the opinion of others only in a multi-period economy. Since agents do not observe the average opinion directly, their private signals have a dual role. The signal informs each agent about the final value, V , and also about the average opinion, $\int_i S^i di$. Agents also use the information in intermediate prices to learn about the opinions of other agents.

2) There is aggregate noise in the economy in the form of supply shocks. The noisy supply is given by $Z_t = \sum_{t'=1}^t u_{t'}$, where $u_{t'} \sim N(0, 1/\rho_u)$ are independent over time.

The presence of aggregate noise Z_t is, of course, standard in the noisy REE models (see Hellwig (1980), Grossman (1976)). However, its role here is somewhat different than usual. In a REE model, noise is used to facilitate trade which otherwise may be precluded due to standard “No-Trade Theorem” arguments. Since we have difference of opinions, agents trade even without noisy supply. In our model, noise ensures that the process by which agents learn about the opinions of other agents is not degenerate. In the class of linear equilibria we characterize next, no noise implies that agents can infer from prices precisely what other agents think. This would eliminate the kind of price and learning dynamics we are interested in.

Agents maximize their utility over final wealth:

$$u(W) = -e^{-\gamma W} \tag{6}$$

All agents start with zero wealth / endowment so their final wealth W is the sum of their trading profits $W = \sum_{t < T} w_t$ where w_t denotes the capital gain in their portfolio from time t to $t + 1$. Note that the last period is identical to the static case. Hence, one would get a drift under the same conditions described in the previous section. We focus on the drift in the earlier periods. Specifically, we say that there is a *price drift* if $E[P_2 - P_1 | P_1]$ increases in P_1 .

3.2 Simple DOO structure

We first consider a model with only first order difference of opinions. Agents agree to disagree about the value of the asset itself but not about what the average opinion is. Agents hold different beliefs about the average opinion since their beliefs about the value of the asset serve as a private signal about the average opinion. Given our structure each agent believes that the average opinion is normally distributed with mean $E^i[V]$.

We solve the model using backward induction. In the last round of trading, agents trade only based on their assessment of the mean $E^i[V]$ and the precision ρ_V . Using our CARA utility assumption combined with the normal distribution we get that at time $t = 2$, optimal demand of the agent is given by

$$x_2^i(P_2, E^i[V]) = \frac{\rho_V (E^i[V] - P_2)}{\gamma}. \quad (7)$$

Aggregating over the demand of all agents and equating demand to supply then yields

$$P_2 = \bar{E}[V] - \frac{\gamma}{\rho_V} Z_2. \quad (8)$$

The main technical challenge is the fact that agents have hedging demands at $t = 1$. At time $t = 1$ the agent optimizes his utility over first round trading profits taking into account the profits he expects to gain in the second round. For the current round we first conjecture a linear equilibrium, that is

$$P_1 = a\bar{E}[V] + bZ_1. \quad (9)$$

Based on this conjecture we have that conditional on P_1 and $E^i[V]$ the price in the next period, P_2 , is normally distributed

$$P_2 \stackrel{i}{\sim} N\left(E^i[P_2 | P_1, E^i[V]], 1 / \rho_{P_2}\right) \quad (10)$$

Moreover, the conditional mean is linear so that

$$E^i [P_2 | P_1, E^i[V]] = k_1 P_1 + k_2 E^i[V]. \quad (11)$$

Thus we can describe the demand function of agent i at time $t = 1$ as

$$x_1^i(P_1, E^i[V]) = \arg \max_x E^i \left[-e^{-\gamma W} | P_1, E^i[V] \right], \quad (12)$$

such that $W = w_1 + w_2 = x(P_2 - P_1) + w_2$. Based on (7) and (8) we have that agent i 's trading profits from the next trading period are given by:

$$w_2 = x_2^i(P_2, E^i[V])(V - P_2) = \frac{1}{\gamma} \rho_V (E^i[V] - P_2)(V - P_2) \quad (13)$$

Given that agents have a difference of opinions about the final value of the asset, we show the following result in the appendix.

Proposition 2 (i) *A unique linear equilibrium described by (11) exists in which $P_1 = a\bar{E}[V] + bZ_1$ and $E^i [P_2 | P_1, E^i[V]] = k_1 P_1 + k_2 E^i[V]$.³*

(ii) *Optimal demand in the first period is of the form:*

$$x_1^i (P_1, E^i[V]) = \frac{1}{\gamma} \rho_V (E^i[V] - P_1) + \frac{1}{\gamma} \rho_{P_2} (E^i [P_2 | P_1, E^i[V]] - P_1) \quad (14)$$

As a direct consequence of the difference in opinions, there are hedging demands faced by the agent. This implies that the optimal demand has an interesting form consisting of two components:

- 1) Speculative Position – This is the position that the agent undertakes as a result of speculating on the next period's price: $\frac{1}{\gamma} \rho_{P_2} (E^i [P_2 | P_1, E^i[V]] - P_1)$
- 2) Long Term Position – This is the position that the agent undertakes as a result of his long term view of the asset's final value: $\frac{1}{\gamma} \rho_V (E^i[V] - P_1)$

The long term position reflects the difference of opinions assumption. Agents base this component of their trade on their view of the fundamental value of the asset. The speculative position reflects speculations on what the other agents' valuations are. We shall examine how this pattern results in positive autocorrelation in this and the next subsection.

³ The explicit characterization of a, b, k_1 , and k_2 appears in the appendix at the end of the proof of the proposition.

As the final result in this part, we note that just a first order difference in opinions is not enough to generate drift in prices. We show the following result in the appendix.

Proposition 3 *Without higher order difference of beliefs, there is no price drift in the prices between periods 1 and 2, i.e., $E[P_2 - P_1 | P_1] = 0$.*

The reason is that the agents behave rationally when trying to infer the average opinion for the intermediate period. Since they do not exhibit higher order difference in beliefs, agents update their beliefs about the average valuation using the price, just as agents in an REE model learn about the final value. This aligns the beliefs about the price in the future with the current price, and so there is no drift.

3.3 Higher Order Difference of Beliefs

We next consider the case where there is also second order difference of opinion. Each agent is now dogmatic both in his views on what he thinks will be the terminal value of the asset, and in his beliefs about the distribution of the average beliefs of all agents. In addition to the setup in the previous section, agent i receives a second signal $T^i = \bar{E}[V] + \eta^i$, where $\eta^i \sim N(0, 1/\rho_\eta)$ and independent across agents. As a result, agent i 's posterior beliefs about the average valuation $\bar{E}[V]$ is normally distributed with mean $E^i[\bar{E}[V]]$, where⁴

$$E^i[\bar{E}[V]] = \frac{\rho_{\bar{V},0}}{\rho_{\bar{V},0} + \rho_\eta} E^i[V] + \frac{\rho_\eta}{\rho_{\bar{V},0} + \rho_\eta} T^i \quad (15)$$

We denote the average expectation across agents of the average valuation by

$$\bar{E}[\bar{E}[V]] = \int_i E^i[\bar{E}[V]] di. \quad (16)$$

In this set up we assume that agents have a second order difference of opinions. Specifically, agent i believes that given $E^i[\bar{E}[V]]$, the true value of $\bar{E}[V]$ is independent of $\bar{E}[\bar{E}[V]]$. This implies that agent i believes the average beliefs of the average valuation $\bar{E}[\bar{E}[V]]$ is normally distributed with mean $E^i[\bar{E}[V]]$.

⁴ The precision of these beliefs is described in the appendix.

In the last round of trading second order beliefs do not matter, so agents trade only based on their assessment of the mean $E^i[V]$ and the precision ρ_V . As a result the price function at time 2 is the same as it was for the case where agents had only first order disagreement. In particular, it takes the form:

$$P_2 = \bar{E}[V] - \gamma Z_2 / \rho_V.$$

We conjecture that $P_1 = a\bar{E}[V] + bZ_1 + c\bar{E}[\bar{E}[V]]$. Based on this conjecture we have that conditional on P_1 , $E^i[V]$, $E^i[\bar{E}[V]]$ the price in the next period, P_2 , is normally distributed:

$$P_2 \stackrel{i}{\sim} N\left(E^i\left[P_2 \mid P_1, E^i[V], E^i[\bar{E}[V]]\right], 1 / \rho_{P_2}\right).$$

Each agent believes that conditional on observing the signals S^i and T^i , $\bar{E}[V]$ and $\bar{E}[\bar{E}[V]]$ are independent so that

$$P_1 \stackrel{i}{\sim} N\left(aE^i[V] + cE^i[\bar{E}[V]], 1 / \rho_{P_1}\right) \quad (17)$$

Agents demand functions can be derived similar to the derivation of Proposition 2, where $E^i\left[P_2 \mid P_1, E^i[V]\right]$ is replaced by $E^i\left[P_2 \mid P_1, E^i[V], E^i[\bar{E}[V]]\right]$ so that we get

$$x_1^i\left(P_1, E^i[V], E^i[\bar{E}[V]]\right) = \frac{1}{\gamma} \rho_V \left(E^i[V] - P_1\right) + \frac{1}{\gamma} \rho_{P_2} \left(E^i\left[P_2 \mid P_1, E^i[V], E^i[\bar{E}[V]]\right] - P_1\right) \quad (18)$$

Moreover, the equilibrium in this set up is characterized by the following proposition.

Proposition 4

(i) $P_1 = a\bar{E}[V] + bZ_1 + c\bar{E}[\bar{E}[V]],$

$$E^i\left[P_2 \mid P_1, E^i[V], E^i[\bar{E}[V]]\right] = k_1 P_1 + k_2 E^i[V] + k_3 E^i[\bar{E}[V]],^5$$

(ii) *With higher order difference of beliefs, there is price drift between periods 1 and 2, i.e., $E\left[P_2 - P_1 \mid P_1\right]$ increases in P_1 .*

3.4 Discussion

As in much of the previous literature, our model is concerned with a single risky asset. While our results fit the serial correlation in returns at the aggregate market level, we prefer to interpret them in terms of the price drift or momentum effect in the cross

⁵ The explicit characterization of a, b, c, k_1, k_2 and k_3 appears in the appendix at the end of the proof of the first part of the proposition.

section of individual stocks. This raises two potential concerns. If the results are driven by risk aversion, the relevant risk in a single asset may be significantly lower due to diversification. Secondly, most of the empirical evidence on momentum considers the cross section of returns rather than serial correlation of individual stocks. Specifically, these tests focus on a portfolio in which there is a long position of past ‘winners’ and a short position in ‘losers’.

The first issue is not a concern in our model. The effect of lower risk due to diversification would imply a lower risk aversion coefficient. In the absence of aggregate supply noise, prices are not affected by the risk aversion coefficient. Moreover, when there is aggregate noise, the autocorrelation is decreasing in risk aversion. Hence, as long as agents are not risk neutral, our conclusion holds. The same qualitative observation can be made in the dynamic model.

To address the second issue, we solve a multi-asset counterpart of the static difference of opinion model in the appendix.⁶ With multiple assets, we define momentum in the cross section by requiring that for every two risky assets i and j , $P_1^i > P_1^j$ implies that $E[P_2^i - P_1^i | P_1] > E[P_2^j - P_1^j | P_1]$, where P_t is the price vector of all securities at date t , and P_t^m is the price of security m at date t . Under the assumption that cash flows are identically distributed we show that without noisy supply shocks there always exists momentum. With noisy supply shocks, there exists momentum in the cross section, under some parameter restrictions. For example, if the risky cash flows, supply shocks, and signals are all independent and identically distributed across assets, then the parameter restriction that implies positive serial autocorrelation in a model with one stock obviously implies momentum at the cross section with multiple assets.

Finally, we note that our objective has been to evaluate whether slow aggregation of information/opinions can lead to price drift. To get a comprehensive understanding of the matter at hand, we have analyzed both REE and DOO settings. The philosophical debate about which of these paradigms is more appropriate has been fierce over the

⁶ The multi-asset REE counterpart appears in Admati (1985).

years, but has largely remained inconclusive.⁷ A part of this debate has centered on whether DOO models can be supported in a stationary, repeated setting, where investors can learn how to use prices over time. This issue is beyond the scope of our model, which is stylized and has a finite horizon. However, recently Acemoglu, Chernozhukov and Yildiz (2006) have shown that in an environment in which individuals are uncertain about how they should interpret signals, individuals with different priors may never agree, even after observing the same infinite sequence of signals.

We do not have a stake in the REE/DOO debate at large, and have simply analyzed the feasibility of the proposed mechanism under both approaches. For supporters of the REE models, our paper shows that the slow aggregations of information is not an appropriate channel to generate price drift. For supporters of DOO models, our analysis demonstrates that, conceptually, DOO models may generate price drift through a slow aggregation of opinions channel. However, we have also highlighted the fact that relaxing the common knowledge assumption is a necessary condition. Without differences in higher order beliefs, DOO models fail to generate price drift as well.

4 Related Literature

Our paper is most closely related to Allen, Morris and Shin (2006) who consider price drift in a REE model, as a result of higher order beliefs. The starting point for both papers is the fact that in a dynamic model, higher order beliefs become relevant. However we arrive at very different conclusions regarding the implications for the resulting price patterns. AMS argue that higher order beliefs may lead to price drift in REE models. We argue that since the public information available to agents includes the price in a REE, such patterns do not arise. Moreover, with noise in the economy, prices instead exhibit negative autocorrelation. We have shown that only models in

⁷A discussion of the relevance of the non common prior assumption appears in Morris (1995), for example.

which agents agree to disagree can make the AMS intuition valid as agents put less weight on prices.

To see why we get a different result from AMS, begin by considering the example that appears in the first part of their paper. This example presents a statistical exercise that provides the intuition for the REE model. Agents are interested in estimating a random variable V . All agents have common priors about the distribution of V , given by $V \sim N(0, 1/\rho_V)$. Agents observe two signals: (i) a public signal, Y , and, (ii) a private signal S^i . The signals are normally distributed, with $Y = V + \delta$ and $S^i = V + \varepsilon^i$ where $\{\varepsilon^i\}$ are i.i.d. and independent of δ ; the distributions are given by $\varepsilon^i \sim N(0, 1/\rho_\varepsilon)$ and $\delta \sim N(0, 1/\rho_\delta)$. In this case, we have

$$E^i[V] \equiv E[V | S^i, Y] = \frac{\rho_\varepsilon}{\rho_\varepsilon + \rho_\delta + \rho_V} S^i + \frac{\rho_\delta}{\rho_\varepsilon + \rho_\delta + \rho_V} Y + \frac{\rho_V}{\rho_\varepsilon + \rho_\delta + \rho_V} 0$$

Hence,

$$\bar{E}[V] \equiv \int E^i[V] = \frac{\rho_\varepsilon}{\rho_\varepsilon + \rho_\delta + \rho_V} \bar{S} + \frac{\rho_\delta}{\rho_\varepsilon + \rho_\delta + \rho_V} Y = \frac{\rho_\varepsilon}{\rho_\varepsilon + \rho_\delta + \rho_V} V + \frac{\rho_\delta}{\rho_\varepsilon + \rho_\delta + \rho_V} Y$$

where the second equality follows from the fact that with a continuum of agents we have that $\bar{S} = V$.⁸

Hence, the average estimate is indeed biased as it puts excessive weight on the public signal Y . As AMS note, this feature breaks the martingale property and may generate positive autocorrelation or price drifts. If one defines $X_T = V$, and $X_t = \bar{E}[X_{t+1}]$, then one can show that $X_t = \alpha_t Y + \beta_t V$ where α_t is decreasing in t and β_t is increasing in t . Given this representation, it is tempting to conclude that similar reasoning may lead to drift in a rational expectation equilibrium. The reason why we argue this is not the case is that in a REE the public signal is the price of the asset. Agents observe the price which reflects their average opinion and use it to correct any bias. In terms of the example above this implies that Y is a function of the X_t and hence one cannot simply look at the dynamics of α_t and β_t . The simple model we have examined in section 2.1 demonstrates this effect in a static setup.

⁸ The exact structure in AMS is a slightly reduced version of this example. Specifically, $V \sim N(y, 1/\rho_V)$, where y the ex ante mean of V is exogenously specified.

In the second part of AMS, they consider a standard REE model in which they argue that a price drift arises; they base this claim on Propositions 1 and 2 of their paper. While they do not formally define price drift, following Proposition 2 they suggest that a situation in which the price approaches the true fundamental value in incremental steps would have many outward appearances of momentum in prices. To see why conditioning on the ex-post realized final value is problematic, consider a price process that is a martingale; the price in each period equals the expected value of the asset conditional on current information. If the information sets are increasing over time, then conditional on the final value, we find that the price slowly converges to this value. Still, such a price process does not exhibit a price drift; since it is a martingale, there is no autocorrelation or predictability in returns. Moreover, many standard information based models exhibit such behavior. In particular, the price process in the multi-period model in Kyle (1985) and in Glosten and Milgrom (1985) satisfy propositions 1 and 2 in AMS.

Our paper is also closely related to the “beauty contest” metaphor from Keynes (1936). Keynes based his analogy on contests that were popular in England at the time, where a newspaper would print 100 photographs, and people would write in and say which six faces they liked most. Everyone who picked the most popular face was automatically entered in a raffle, where they could win a prize. Given these incentives, people would not necessarily choose faces they found the prettiest, but instead choose those they believed would catch the fancy of the other competitors, all of whom were using the same logic in making their choices. This led agents to form higher order beliefs. The link to financial markets follows from the fact that prices reflect some average opinion of different investors. Since it is possible to resell the stock, it may not be enough for an investor to pick the stock he finds most attractive, as he must also consider which stocks others will find attractive. As a result, investors need to form beliefs about the average valuation, the average opinion about the average valuation, and so on; in doing so they engage in higher order reasoning. Indeed, AMS show formally in a finite horizon economy that the price t periods before the last one reflects the t -th order average

opinion. One may argue that the Keynes' intuition is more appropriate in a DOO model, however, since agents have different fundamental valuations. In contrast, investors in a REE model agree that there is an objective fundamental value and cannot agree to disagree, even if they have different information. Nevertheless, the main motivation we provide for a model with differences in higher order beliefs is that it generates a price drift.

Other papers such as Makarov and Rytchkov (2006), consider models of asymmetric information in which non-trivial higher order beliefs potentially lead to possible correlations in price changes. However, in Makarov and Rytchkov (2006), positive serial auto-correlation arises only under a very specific noise structure which is correlated itself. Hence, in their model, the heterogeneity of beliefs is not the component that generates the drift in prices.⁹

While we focus on the heterogeneity of beliefs as a potential explanation, a number of alternative explanations for price drifts have been proposed. Broadly, they fall into two categories: behavioral and rational. In behavioral models, some or all agents in the economy exhibit specific cognitive biases that lead to under-reaction. For example, in Barberis, Shleifer and Vishny (1998), agents exhibit a conservatism bias, while in Daniel, Hirshleifer and Subrahmanyam (1998), agents overestimate the precision of their signals and suffer from a self attribution bias. Hong and Stein (1999) assume the presence of "news-watchers" who receive public signals slowly, but do not use the price to update their beliefs. These behavioral biases lead to an under-reaction to public information, and so lead to price drift or momentum.

The rational explanations are either risk-based or information-based. In risk-based models the drift is a result of the dynamics of the underlying fundamentals. In Berk,

⁹ Kondor (2004) shows, in a rational expectations framework, how differences in higher order beliefs can lead to high trading volume and volatile prices following public announcements. Cao and Ou-Yang (2005) demonstrate, in a difference of opinions setting, how failure of the law of iterated expectations for average beliefs can cause prices to be higher (lower) than the price that any individual trader would be willing to pay for the asset if she was precluded to trade again in subsequent periods.

Green and Naik (1999), the potential driving force is variation of exposure over the life-cycle of a firm’s endogenously chosen investment projects.¹⁰ In Johnson (2002), momentum potentially arises as an artifact of stochastic expected growth rates of a firm’s cash flows. In Holden and Subrahmanyam (2002), momentum is a result of increased precision of information over time. Sequential arrival of information prior to the terminal date leads to a decrease in the risk borne by the market because the mass of informed traders increases over time. As a consequence, there is a gradual decrease in the conditional risk premium required to absorb liquidity shocks.

As we have shown, to potentially generate price drift in a DOO setting through slow aggregation of opinions channel one needs to depart from the common knowledge assumption that is typically made in DOO models. Biais and Bossaerts (1998) show how one can relax the common knowledge assumption and still avoid the infinite regress problem. Varian (1989) considers a fully revealing equilibrium in which agents have different priors and receive subsequent information. Ottaviani and Sorensen (2006) analyze para-mutual betting markets with heterogeneous priors and private information. Allen, Morris and Postlewaite (1993) have both differences in priors and differences in information interacting. In their model, a necessary condition for a strong bubble to occur is that agents’ trades are not common knowledge.

5 Conclusion

We have analyzed the potential link between heterogeneous beliefs and price drift. We have demonstrated that in generating price drift or momentum in asset prices through a channel of slow aggregation of heterogeneous beliefs differences in opinions about higher order beliefs are an important factor. Standard models of difference in opinions assume common knowledge of other agents’ opinions – each agent knows what the others believe about the fundamental value of the asset, and literally “agree to disagree”. We depart from the literature by relaxing the assumption of common knowledge, and

¹⁰ Gomes, Kogan and Zhang (2003) relax the partial equilibrium restrictions in Berk, Green, and Naik (1999) analyzing a related problem in general equilibrium.

assume that agents are uncertain about the beliefs of others. Interestingly, we show that in a dynamic framework, with only first order differences in beliefs there is typically no price drift, and prices satisfy a martingale property apart from the last period. This is because in earlier rounds of trade, the equilibrium is similar to a classic REE, except investors infer the opinions of others from the price instead of their private information. To obtain a price drift in this setup, one also needs differences in opinions about higher order beliefs.

We also highlight the fact in a pure rational expectations equilibrium, in which all agents update their beliefs about the value using the price, there can be no price drift. The intuition presented in AMS and others that relates higher order beliefs to price drift holds only when agents' public information is completely exogenous, and in particular, does not include prices. However, in REE models the price is in the agents' information sets, and is endogenously determined to reflect the average opinion of the agents. As a result, a price drift does not arise in REE models contrary to what has been suggested in the literature.

Relaxing the common knowledge assumption in differences of opinion models can potentially have important implications for the dynamics of trading volume, and the dynamic interaction between trading volume and returns. We are currently exploring these aspects in a separate paper.

6 Appendix

Notation: Throughout the appendix the precision of random variable X given information at time t is denoted by $\rho_{X,t}$. If the posterior precision does not change over time, then it is denoted by ρ_X for notational simplicity.

Proof for Proposition 1:

(i) For the REE case,

$$\begin{aligned} E[V - P_1 | P_1] &= E\left[V - \bar{E}[V] + \frac{\gamma}{\rho_V} Z_1 \middle| P_1\right] \\ &= \int_i E\left[V - E^i[V | S^i, P_1] \middle| P_1\right] + \frac{\gamma}{\rho_V} E[Z_1 | P_1] \\ &= \frac{\gamma}{\rho_V} E[Z_1 | P_1] = \frac{\gamma}{\rho_V} k P_1 \end{aligned}$$

for some $k < 0$. The third equality follows from the fact that since agents correctly incorporate the current price in forming beliefs about the future price, and prices are linear, the law of iterated expectation holds for every agent, and averaging over all i does not change this.

(ii) For the DOO case, the price takes the form:

$$P_1 = \bar{E}[V] - \frac{\gamma}{\rho_V} Z_1 = \frac{\rho_\varepsilon}{\rho_\varepsilon + \rho_{V,0}} V - \frac{\gamma}{\rho_\varepsilon + \rho_{V,0}} Z_1$$

Let the noise be distributed normally: $Z_1 \sim N(0, 1/\rho_Z)$. Then, we know

$$\begin{aligned} E[V - P_1 | P_1] &= E[V | P_1] - P_1 \\ &= \frac{\frac{\rho_\varepsilon}{\rho_\varepsilon + \rho_{V,0}} \frac{1}{\rho_{V,0}}}{\left(\frac{\rho_\varepsilon}{\rho_\varepsilon + \rho_{V,0}}\right)^2 \frac{1}{\rho_{V,0}} + \left(\frac{\gamma}{\rho_\varepsilon + \rho_{V,0}}\right)^2 \frac{1}{\rho_Z}} P_1 - P_1 \\ &= \left(\frac{\rho_Z \rho_\varepsilon (\rho_\varepsilon + \rho_{V,0})}{\rho_\varepsilon^2 \rho_Z + \gamma^2 \rho_{V,0}} - 1\right) P_1 \\ &= \left(\frac{\rho_{V,0} (\rho_Z \rho_\varepsilon - \gamma^2)}{\rho_\varepsilon^2 \rho_Z + \gamma^2 \rho_{V,0}}\right) P_1 \end{aligned}$$

QED

Proof for Proposition 2:

We first focus on (ii) assuming a linear equilibrium exists. Consider the optimization problem the agent is facing as given by (12). The fact that the trading profits across the different rounds are not independent implies that it can not be simplified to a myopic optimization problem; that is, agent i has hedging demands. In REE this is not the case, since trading profits in future periods are conditionally independent – hence, no hedging demands arise in the case of a pure rational expectations equilibrium. Using simple algebraic manipulation it can be written as:

$$x_1^i(P_1, E^i[V]) = \arg \max_x E^i \left[-\exp \{ -\gamma x (P_2 - P_1) \} \exp \{ -\rho_V (E^i[V] - P_2)(V - P_2) \} \mid P_1, E^i[V] \right]$$

Furthermore, note that conditional on $E^i[V]$, we know that P_2 (which depends on $\bar{E}[V]$ and Z_2) and V are independent. Hence, we can re-write the optimization problem as:

$$x_1^i(P_1, E^i[V]) = \arg \max_x E^i \left[\exp \{ -\gamma x (P_2 - P_1) \} E^i \left[\exp \{ -\rho_V (E^i[V] - P_2)(V - P_2) \} \mid P_2, P_1, E^i[V] \right] \mid P_1, E^i[V] \right]$$

and note that:

$$\begin{aligned} E^i \left[\exp \{ -\rho_V (E^i[V] - P_2)(V - P_2) \} \mid P_1, E^i[V], P_2 \right] &= \exp \left\{ -\rho_V (E^i[V] - P_2)^2 + \frac{1}{2} \rho_V^2 \frac{1}{\rho_V} (E^i[V] - P_2)^2 \right\} \\ &= \exp \left\{ -\frac{\rho_V}{2} (E^i[V] - P_2)^2 \right\} \end{aligned}$$

which implies that the original optimization problem is now given by:

$$\begin{aligned} x_1^i(P_1, E^i[V]) &= \arg \max_x E^i \left[\exp \left\{ -\gamma x (P_2 - P_1) - \frac{\rho_V}{2} (E^i[V] - P_2)^2 \right\} \mid P_1, E^i[V] \right] \\ &= \arg \max_x E^i \left[\exp \left\{ -\frac{\rho_V}{2} P_2^2 + (-\gamma x + \rho_V E^i[V]) P_2 + \gamma x P_1 - \frac{\rho_V}{2} E^i[V]^2 \right\} \mid P_1, E^i[V] \right] \end{aligned}$$

Note that this expression is that of the moment generating function for the quadratic form of a normal. We apply the following standard result.

Lemma: If $\xi \sim N(\mu_\xi, \frac{1}{\rho_\xi})$ then

$$\begin{aligned} &E \left[-\exp \{ c + d\xi + e\xi^2 \} \right] \\ &= - \left| 1 - 2 \frac{e}{\rho_\xi} \right|^{-1} \exp \left(-\frac{1}{2} (\mu_\xi^2 \rho_\xi - 2c) + \frac{1}{2} \left(\mu_\xi + \frac{d}{\rho_\xi} \right) \left(1 - 2 \frac{e}{\rho_\xi} \right)^{-1} \rho_\xi \left(\mu_\xi + \frac{d}{\rho_\xi} \right) \right) \end{aligned}$$

We apply this to our setting, in which $\xi = P_2$, $c = \gamma x P_1 - \frac{\rho_V}{2} E^i[V]^2$, $d = \rho_V E^i[V] - \gamma x$, $e = -\frac{\rho_V}{2}$, $\mu_\xi = E^i[P_2 | P_1, E^i[V]]$, and $\rho_\xi = \rho_{P_2}$. Note that only c and d depend on x .

Hence, the first order condition of the optimization problem with respect to x is:

$$\gamma P_1 - \frac{\gamma \left[(\rho_V E^i[V] - \gamma x) / \rho_{P_2} + E^i[P_2 | P_1, E^i[V]] \right]}{1 + \rho_V / \rho_{P_2}} = 0$$

and this implies that the optimal demand is given by:

$$x_1^i(P_1, E^i[V]) = \frac{1}{\gamma} \rho_V (E^i[V] - P_1) + \frac{1}{\gamma} \rho_{P_2} (E^i[P_2 | P_1, E^i[V]] - P_1).$$

Using the above proposition we can derive the price in the first period using the market clearing condition:

$$\int_i \left[\frac{1}{\gamma} \rho_V (E^i[V] - P_1) + \frac{1}{\gamma} \rho_{P_2} (E^i[P_2 | P_1, E^i[V]] - P_1) \right] di = Z_1$$

Note that the conditional mean of next period's price is linear:

$$E^i[P_2 | P_1, E^i[V]] = k_1 P_1 + k_2 E^i[V].$$

This implies that

$$\frac{1}{\gamma} \rho_V (\bar{E}[V] - P_1) + \frac{1}{\gamma} \rho_{P_2} (k_1 P_1 + k_2 \bar{E}[V] - P_1) = Z_1$$

which implies that the period 1 price is given by:

$$P_1 = \frac{\rho_V + \rho_{P_2} k_2}{\rho_V + (1 - k_1) \rho_{P_2}} \bar{E}[V] - \frac{\gamma}{\rho_V + (1 - k_1) \rho_{P_2}} Z_1$$

Denote investor i 's beliefs about the average valuation as $\bar{E}[V] \stackrel{i}{\sim} N(E^i[V], \rho_{\bar{V},0})$. Also,

using the notation $P_1 = a\bar{E}[V] + bZ_1$, so that $P_1 \stackrel{i}{\sim} N\left(aE^i[V], \frac{a^2}{\rho_V} + \frac{b^2}{\rho_{Z_1,0}}\right)$ we have the

following:

$$E^i[\bar{E}[V] | P_1, E^i[V]] = \frac{\rho_{\bar{V},0}}{\rho_{\bar{V},0} + \frac{a^2}{b^2} \rho_{Z_1,0}} E^i[V] + \frac{\frac{a^2}{b^2} \rho_{Z_1,0}}{\rho_{\bar{V},0} + \frac{a^2}{b^2} \rho_{Z_1,0}} \frac{P_1}{a} = \frac{b^2 \rho_{\bar{V},0}}{b^2 \rho_{\bar{V},0} + a^2 \rho_{Z_1,0}} E^i[V] + \frac{a \rho_{Z_1,0}}{b^2 \rho_{\bar{V},0} + a^2 \rho_{Z_1,0}} P_1$$

$$E^i[Z_2 | P_1, E^i[V]] = \frac{\frac{b^2}{a^2} \rho_{\bar{V},0}}{\frac{b^2}{a^2} \rho_{\bar{V},0} + \rho_{Z_1,0}} \left(\frac{P_1}{b} - \frac{a}{b} E^i[V] \right) = \frac{b \rho_{\bar{V},0}}{b^2 \rho_{\bar{V},0} + a^2 \rho_{Z_1,0}} (P_1 - a E^i[V])$$

So that

$$E^i \left[P_2 \left| P_1, E^i[V] \right. \right] = E \left[\bar{E}[V] \mid P_1, E^i[V] \right] - \frac{\gamma}{\rho_V} E^i \left[Z_2 \mid P_1, E^i[V] \right] = \frac{a\rho_{Z_{1,1}} + \gamma b}{b^2 \rho_{\bar{V},0} + a^2 \rho_{Z_{1,0}}} P_1 - \frac{b^2 \rho_{\bar{V},0} - \gamma b a}{b^2 \rho_{\bar{V},0} + a^2 \rho_{Z_{1,0}}} E^i[V]$$

$$\rho_{Z_{1,0}} = \rho_u, \quad \rho_{Z_{1,1}} = \frac{1}{\text{Var}^i \left[Z_1 \mid P_1, E^i[V] \right]} = b^2 \rho_{\bar{V},0} / a^2 + \rho_u,$$

$$\frac{1}{\rho_{Z_{2,1}}} = \text{Var}^i \left[Z_2 \mid P_1, E^i[V] \right] = \frac{1}{b^2 \rho_{\bar{V},0} / a^2 + \rho_u} + \frac{1}{\rho_u}, \quad \rho_{\bar{V},1} = \rho_{\bar{V},0} + a^2 \rho_u / b^2$$

$$\sigma_{\bar{E}[V], Z_1, 2} = \text{Cov}^i(\bar{E}[V], Z_1 \mid P_2, E^i[V]) = \frac{-ab}{b^2 \rho_V + a^2 \rho_u}$$

$$\frac{1}{\rho_{P_2}} = \frac{1}{\rho_{\bar{V},1}} + \frac{\gamma^2}{\rho_V^2} \frac{1}{\rho_{Z_{2,1}}} + 2 \frac{\gamma}{\rho_V} \sigma_{\bar{E}[V], Z_1, 1}^2$$

$$= \frac{1}{\rho_{\bar{V},0} + a^2 \rho_u / b^2} + \frac{\gamma^2}{\rho_V^2} \left(\frac{1}{b^2 \rho_{\bar{V},0} / a^2 + \rho_u} + \frac{1}{\rho_u} \right) - 2 \frac{\gamma}{\rho_V} \frac{ab}{b^2 \rho_{\bar{V},0} + a^2 \rho_u}$$

$$= \frac{(b - \gamma a / \rho_V)^2}{b^2 \rho_{\bar{V},0} + a^2 \rho_u} + \frac{\gamma^2}{\rho_V^2} \frac{1}{\rho_u}$$

This leaves the following system of equations for a, b, k_1, k_2 , and ρ_{P_2} which characterize the equilibrium:

$$a = \frac{\rho_V + \rho_{P_2} k_2}{\rho_V + (1 - k_1) \rho_{P_2}}, \quad b = \frac{-\gamma}{\rho_V + (1 - k_1) \rho_{P_2}}, \quad k_1 = \frac{a \rho_u + \gamma b}{b^2 \rho_{\bar{V},0} + a^2 \rho_u}, \quad k_2 = -\frac{b^2 \rho_{\bar{V},0} - \gamma b a}{b^2 \rho_{\bar{V},0} + a^2 \rho_u}$$

$$\text{and } \frac{1}{\rho_{P_2}} = \frac{(b - \gamma a / \rho_V)^2}{b^2 \rho_{\bar{V},0} + a^2 \rho_u} + \frac{\gamma^2}{\rho_V^2} \frac{1}{\rho_u}. \quad \text{QED}$$

Proof for Proposition 3:

Note that the price at date 1 can be re-written as:

$$P_1 = \frac{\rho_V}{\rho_V + \rho_{P_2}} \bar{E}[V] + \frac{\rho_{P_2}}{\rho_V + \rho_{P_2}} \bar{E}[P_2 | P_1, E^i[V]] - \frac{\gamma}{\rho_V + \rho_{P_2}} Z_1$$

This implies that expected price changes are zero:

$$\begin{aligned} E[P_2 - P_1 | P_1] &= E \left[P_2 - \frac{\rho_V}{\rho_V + \rho_{P_2}} \bar{E}[V] - \frac{\rho_{P_2}}{\rho_V + \rho_{P_2}} \bar{E}[P_2 | P_1, E^i[V]] + \frac{\gamma}{\rho_V + \rho_{P_2}} Z_1 \middle| P_1 \right] \\ &= E \left[-\frac{\rho_V}{\rho_V + \rho_{P_2}} \frac{\gamma}{\rho_V} Z_2 + \frac{\rho_{P_2}}{\rho_V + \rho_{P_2}} (P_2 - \bar{E}[P_2 | P_1, E^i[V]]) + \frac{\gamma}{\rho_V + \rho_{P_2}} Z_1 \middle| P_1 \right] \\ &= \frac{\gamma}{\rho_V + \rho_{P_2}} E[Z_1 - Z_2 | P_1] \\ &= 0 \end{aligned}$$

The third equality follows from the fact that since the agent's rationally use the information in prices to form their beliefs about $\bar{E}[V]$ and Z_2 , the law of iterated expectations holds for each agent i :

1. $E \left[\left(\bar{E}[V] - E^i[\bar{E}[V] | P_1, E^i[V]] \right) \middle| P_1 \right] = 0,$
2. $E \left[E^i[Z_2 | P_1, E^i[V]] \middle| P_1 \right] = E[Z_2 | P_1] = E[Z_1 | P_1].$

The result follows since all the expectations are linear. QED

Proof for Proposition 4:

The proof of part (i) follows similar to the derivation of the proof of Proposition 2. Let agent i 's beliefs about the average valuation be given by $\bar{E}[V] \stackrel{i}{\sim} N(E^i[\bar{E}[V]], 1/\rho_{\bar{V}})$, where $\rho_{\bar{V}} = \rho_{\bar{V},0} + \rho_{\eta}$. Note that the conditional mean of next period's price is linear:

$$E^i \left[P_2 | P_1, E^i[V], E^i[\bar{E}[V]] \right] = k_1 P_1 + k_2 E^i[V] + k_3 E^i[\bar{E}[V]].$$

Aggregating over investors this implies that

$$\frac{1}{\gamma} \rho_V (\bar{E}[V] - P_1) + \frac{1}{\gamma} \rho_{P_2} (k_1 P_1 + k_2 \bar{E}[V] + k_3 \bar{E}[\bar{E}[V]] - P_1) = Z_1$$

which implies that the period 1 price is given by:

$$P_1 = \frac{\rho_V + k_2 \rho_{P_2}}{\rho_V + (1 - k_1) \rho_{P_2}} \bar{E}[V] - \frac{\gamma}{\rho_V + (1 - k_1) \rho_{P_2}} Z_1 + \frac{\rho_{P_2} k_3}{\rho_V + (1 - k_1) \rho_{P_2}} \bar{E}[\bar{E}[V]]$$

Let agent i 's beliefs about average beliefs about the average valuation be given by $\bar{E}[\bar{E}[V]] \stackrel{i}{\sim} N\left(E^i[\bar{E}[V]], 1/\rho_{\bar{V}}\right)$, and note that $\frac{1}{\rho_R} = \frac{a^2}{\rho_{\bar{V}}} + \frac{b^2}{\rho_{Z_{1,0}}} + \frac{c^2}{\rho_{\bar{V}}}$. Denote by

$$A = \frac{1}{a}(bZ_1 + c\bar{E}[\bar{E}[V]]) \text{ and } B = \frac{1}{b}(a\bar{E}[V] + c\bar{E}[\bar{E}[V]]), \text{ so that } \frac{1}{\rho_A} = \frac{1}{a^2}\left(\frac{b^2}{\rho_{Z_{1,0}}} + \frac{c^2}{\rho_{\bar{V}}}\right) \text{ and}$$

$$\frac{1}{\rho_B} = \frac{1}{b^2}\left(\frac{a^2}{\rho_{\bar{V}}} + \frac{c^2}{\rho_{\bar{V}}}\right). \text{ Using these definitions we get}$$

$$\begin{aligned} E\left[\bar{E}[V] \mid P_1, E^i[V], E^i\bar{E}[V]\right] &= \frac{\rho_V}{\rho_V + \rho_A} E^i[V] + \frac{\rho_A}{\rho_V + \rho_A} \left(\frac{P_1 - cE^i\bar{E}[V]}{a}\right) \\ &= \frac{\rho_V}{\rho_V + a^2 \frac{\rho_{Z_{1,0}}\rho_{\bar{V}}}{b^2\rho_{\bar{V}} + c^2\rho_{Z_{1,0}}}} E^i[V] + \frac{\frac{\rho_{Z_{1,0}}\rho_{\bar{V}}}{b^2\rho_{\bar{V}} + c^2\rho_{Z_{1,0}}}}{\rho_V + \frac{\rho_{Z_{1,0}}\rho_{\bar{V}}}{b^2\rho_{\bar{V}} + c^2\rho_{Z_{1,0}}}} \left(\frac{P_1 - cE^i\bar{E}[V]}{a}\right) \\ E\left[Z_2 \mid P_1, E^i[V], E^i\bar{E}[V]\right] &= \frac{\rho_B}{\rho_B + \rho_{Z_{1,0}}} \left(\frac{P_1 - aE^i[V] - cE^i\bar{E}[V]}{b}\right) = \frac{\frac{\rho_{\bar{V}}\rho_{\bar{V}}}{a^2\rho_{\bar{V}} + c^2\rho_{\bar{V}}}}{\frac{\rho_{\bar{V}}\rho_{\bar{V}}}{a^2\rho_{\bar{V}} + c^2\rho_{\bar{V}}} + \rho_{Z_{1,0}}} \left(\frac{P_1 - aE^i[V] - cE^i\bar{E}[V]}{b}\right) \\ &= \frac{\rho_{\bar{V}}\rho_{\bar{V}}}{\rho_{\bar{V}}\rho_{\bar{V}} + \rho_{Z_{1,0}}(a^2\rho_{\bar{V}} + c^2\rho_{\bar{V}})} \left(\frac{P_1 - aE^i[V] - cE^i\bar{E}[V]}{b}\right) \end{aligned}$$

So that

$$\begin{aligned} E^i\left[P_2 \mid P_1, E^i[V], E^i\bar{E}[V]\right] &= \left(\frac{\rho_A}{\rho_V + \rho_A} \frac{1}{a} - \frac{\rho_B}{\rho_B + \rho_{Z_{1,0}}} \frac{1}{b} \frac{\gamma}{\rho_V}\right) P_1 + \left(\frac{\rho_A}{\rho_V + \rho_A} \frac{1}{a} + \frac{\rho_B}{\rho_B + \rho_{Z_{1,0}}} \frac{1}{b} \frac{\gamma}{\rho_V}\right) aE^i[V] \\ &\quad - \left(\frac{\rho_A}{\rho_V + \rho_A} \frac{1}{a} - \frac{\rho_B}{\rho_B + \rho_{Z_{1,0}}} \frac{1}{b} \frac{\gamma}{\rho_V}\right) cE^i\bar{E}[V] \end{aligned}$$

In addition, keeping in mind that $\rho_{Z_{1,1}} = \rho_u$, we get

$$\rho_{Z_{1,1}} = \frac{1}{\text{Var}^i\left[Z_1 \mid P_1, E^i[V], E^i\bar{E}[V]\right]} = \rho_B + \rho_u, \quad \frac{1}{\rho_{Z_{2,1}}} = \text{Var}^i\left[Z_2 \mid P_1, E^i[V]\right] = \frac{1}{\rho_B + \rho_u} + \frac{1}{\rho_u},$$

$\rho_{\bar{V},1} = \rho_{\bar{V}} + \rho_A$, and

$$\sigma_{\bar{E}[V], Z_{1,1}} = Cov^i(\bar{E}[V], Z_1 | P_1, E^i[V], E^i \bar{E}[V]) = \frac{-ab \frac{1}{\rho_{\bar{V},1}} \frac{1}{\rho_u}}{a^2 \frac{1}{\rho_{\bar{V},1}} + b^2 \frac{1}{\rho_u} + c^2 \frac{1}{\rho_{\bar{V},1}}}$$

$$\text{and } \frac{1}{\rho_{P_2}} = \frac{1}{\rho_{\bar{V},1}} + \frac{\gamma^2}{\rho_V^2} \frac{1}{\rho_{Z_2,1}} + 2 \frac{\gamma}{\rho_V} \sigma_{\bar{E}[V], Z_{1,1}}.$$

Leading to the following system of equations for a, b, c, k_1, k_2 , and k_3 :

$$\begin{aligned} a &= \frac{\rho_V + k_2 \rho_{P_2}}{\rho_V + (1 - k_1) \rho_{P_2}}, \quad b = \frac{-\gamma}{\rho_V + (1 - k_1) \rho_{P_2}}, \quad c = \frac{\rho_{P_2} k_3}{\rho_V + (1 - k_1) \rho_{P_2}}, \\ k_1 &= \left(\frac{\rho_A}{\rho_V + \rho_A} \frac{1}{a} - \frac{\rho_B}{\rho_B + \rho_u} \frac{1}{b} \frac{\gamma}{\rho_V} \right), \quad k_2 = \left(\frac{\rho_A}{\rho_V + \rho_A} \frac{1}{a} + \frac{\rho_B}{\rho_B + \rho_u} \frac{1}{b} \frac{\gamma}{\rho_V} \right) a, \\ k_3 &= - \left(\frac{\rho_A}{\rho_V + \rho_A} \frac{1}{a} - \frac{\rho_B}{\rho_B + \rho_u} \frac{1}{b} \frac{\gamma}{\rho_V} \right) c \\ \frac{1}{\rho_A} &= \frac{1}{a^2} \left(\frac{b^2}{\rho_{Z_1,0}} + \frac{c^2}{\rho_{\bar{V}}} \right), \quad \frac{1}{\rho_B} = \frac{1}{b^2} \left(\frac{a^2}{\rho_{\bar{V}}} + \frac{c^2}{\rho_{\bar{V}}} \right) \end{aligned}$$

The completes the proof of part (i) of the proposition. We next proceed to proving part (ii). Again, prices can be rewritten as:

$$P_1 = \frac{\rho_V}{\rho_V + \rho_{P_2}} \bar{E}[V] + \frac{\rho_{P_2}}{\rho_V + \rho_{P_2}} \bar{E}[P_2 | P_1, E^i[V], E^i \bar{E}[V]] - \frac{\gamma}{\rho_V + \rho_{P_2}} Z_1$$

This implies that expected price changes are given by:

$$\begin{aligned} E[P_2 - P_1 | P_1] &= E \left[\bar{E}[V] - \frac{\gamma}{\rho_V} Z_2 - P_1 \middle| P_1 \right] \\ &= E \left[\frac{\rho_{P_2}}{\rho_V + \rho_{P_2}} \left(\bar{E}[V] - \bar{E}[P_2 | P_1, E^i[V], E^i \bar{E}[V]] \right) - \frac{\gamma}{\rho_V} Z_2 + \frac{\gamma}{\rho_V + \rho_{P_2}} Z_1 \middle| P_1 \right] \\ &= \frac{\rho_{P_2}}{\rho_V + \rho_{P_2}} E \left[\left(\bar{E}[V] - \bar{E}[\bar{E}[V] | P_1, E^i[V], E^i \bar{E}[V]] \right) \middle| P_1 \right] \\ &\quad + E \left[\frac{\rho_{P_2}}{\rho_V + \rho_{P_2}} \frac{\gamma}{\rho_V} \bar{E}[Z_2 | P_1, E^i[V], E^i \bar{E}[V]] - \frac{\gamma}{\rho_V} Z_2 + \frac{\gamma}{\rho_V + \rho_{P_2}} Z_1 \middle| P_1 \right] \end{aligned}$$

We shall solve each expectation separately.

We know that $E^i[P_2 | P_1, E^i[V], E^i \bar{E}[V]] = k_1 P_1 + k_2 E^i[V] + k_3 E^i \bar{E}[V]$,

$P_1 = a\bar{E}[V] + bZ_1 + c\bar{E}\bar{E}[V]$. Also, note that the law of iterated expectations implies: $E[\bar{E}\bar{E}[V] | P_1] = E[E[\bar{E}\bar{E}[V] | P_1, \bar{E}[V]] | P_1] = E[\bar{E}[V] | P_1]$. The first expectation evaluates to

$$\begin{aligned} E\left[\left(\bar{E}[V] - \bar{E}[\bar{E}[V] | P_1, E^i[V], E^i\bar{E}[V]]\right) \middle| P_1\right] &= E\left[\left(\bar{E}[V] - \frac{\rho_V}{\rho_V + \rho_A}\bar{E}[V] - \frac{\rho_A}{\rho_V + \rho_A}\left(\frac{P_1 - c\bar{E}\bar{E}[V]}{a}\right)\right) \middle| P_1\right] \\ &= E\left[\left(\frac{\rho_A}{\rho_V + \rho_A}\bar{E}[V] - \frac{\rho_A}{\rho_V + \rho_A}\left(\bar{E}[V] + \frac{b}{a}Z_1\right)\right) \middle| P_1\right] \\ &= \kappa_1 P_1 \text{ for some } \kappa_1 > 0 \end{aligned}$$

Similarly, the second expectation also evaluates to

$$\begin{aligned} &E\left[\frac{\rho_{P_2}}{\rho_V + \rho_{P_2}} \frac{\gamma}{\rho_V} \bar{E}[Z_2 | P_1, E^i[V], E^i\bar{E}[V]] - \frac{\gamma}{\rho_V} Z_2 + \frac{\gamma}{\rho_V + \rho_{P_2}} Z_1 \middle| P_1\right] \\ &= E\left[\frac{\rho_{P_2}}{\rho_V + \rho_{P_2}} \frac{\gamma}{\rho_V} \bar{E}[Z_1 | P_1, E^i[V], E^i\bar{E}[V]] - \frac{\gamma}{\rho_V} Z_1 + \frac{\gamma}{\rho_V + \rho_{P_2}} Z_1 \middle| P_1\right] \\ &= E\left[\frac{\rho_{P_2}}{\rho_V + \rho_{P_2}} \frac{\gamma}{\rho_V} \frac{\rho_B}{\rho_B + \rho_{Z_{1,0}}} \left(\frac{P_1 - a\bar{E}[V] - c\bar{E}\bar{E}[V]}{b}\right) - \frac{\gamma}{\rho_V} Z_1 + \frac{\gamma}{\rho_V + \rho_{P_2}} Z_1 \middle| P_1\right] \\ &= E\left[\frac{\rho_{P_2}}{\rho_V + \rho_{P_2}} \frac{\gamma}{\rho_V} \frac{\rho_B}{\rho_B + \rho_{Z_{1,0}}} Z_1 - \frac{\gamma}{\rho_V} Z_1 + \frac{\gamma}{\rho_V + \rho_{P_2}} Z_1 \middle| P_1\right] \\ &= \gamma \frac{\rho_{P_2}}{\rho_V + \rho_{P_2}} \frac{1}{\rho_V} \left(\frac{\rho_B}{\rho_B + \rho_{Z_{1,0}}} - 1\right) E[Z_1 | P_1] \\ &= \kappa_2 P_1 \text{ for some } \kappa_2 > 0 \end{aligned}$$

This implies that the conditional expected price changes, $E[P_2 - P_1 | P_1]$, are positively correlated with P_1 . QED

Difference of Opinion with Multiple Assets

With multiple assets V, Z , and ε_i are vectors, so that each investor receives a signal vector S^i :

$$S^i = V + \varepsilon_i, \quad (1)$$

Denote the variance covariance matrices of V, Z , and ε respectively by Σ_V, Σ_Z , and Σ_ε . Bayes rule implies that

$$E[V|S^i] = A_1 S^i \text{ and } \text{var}[V|S^i] = A_2$$

Where the matrices A_1 and A_2 are defined by

$$A_1 = \Sigma_V (\Sigma_V + \Sigma_\varepsilon)^{-1} \quad (2)$$

$$A_2 = \Sigma_V (I - A_1^T) \quad (3)$$

Following similar steps to the ones conducted in the single asset case, we obtain that the price vector P satisfies:

$$P = A_1 V - \gamma A_2 Z \quad (4)$$

So that

$$\begin{aligned} E[V - P | P] &= \Sigma_V A_1^T (A_1 \Sigma_V A_1^T + \gamma^2 A_2 \Sigma_Z A_2^T)^{-1} P - P \\ &= (\Sigma_V A_1^T (A_1 \Sigma_V A_1^T + \gamma^2 A_2 \Sigma_Z A_2^T)^{-1} - I) P \end{aligned}$$

Similar to the one dimensional case, we can define momentum in the cross section by requiring that assets that have higher price today will have in expectation a higher price increase next period.

Whether momentum exists in the cross section depends on the properties of the matrix .

$$H = \Sigma_V A_1^T (A_1 \Sigma_V A_1^T + \gamma^2 A_2 \Sigma_Z A_2^T)^{-1} - I \quad (5)$$

When cashflows are not identically distributed it is evident that one can construct counter examples in which momentum does not hold in the cross section. We therefore restrict the analysis to the case where all cashflows are identically distributed. A similar condition is imposed on the components of the vector of supply shocks Z and the vectors of noisy signal shocks ε_i .¹¹

¹¹Note that identically distributed components implies that all diagonal elements of the variance covariance matrix are the same, and all the off diagonal elements are the same as well.

A sufficient condition for momentum in the cross section is that all diagonal elements of H are the same, all off diagonal elements are the same, and the diagonal elements are larger than the off diagonal ones.¹²

Without noisy supply (i.e., $Z \equiv 0$) the matrix H reduces to

$$H = \Sigma_V^T A_1^T (A_1 \Sigma_V A_1^T)^{-1} - I,$$

and plugging in for A_1 and simplifying yields

$$H = \Sigma_\varepsilon \Sigma_V^{-1}$$

As a result, we have the following lemma.

Lemma 1. *Assume all of cashflows are identically distributed and all signals are also identically distributed. Then if all the of diagonal terms in Σ_ε and Σ_V are the same then prices exhibit momentum.*

Proof. The result follows directly from the fact that the two matrices Σ_ε and Σ_V are positive definite, in each of them all elements on the diagonal are the same positive elements, and all off diagonal elements are the same and are lower in absolute value than the diagonal elements.

Corollary 1. *With two risky assets, with identically distributed cashflows there is always momentum.*

With noisy supply, numerical analysis shows that there exists momentum if Σ_Z and Σ_ε are sufficiently small.

¹²When cashflows are iid and in addition the elements of Z and ε_i are each iid the condition for momentum in the cross section trivially coincides with the condition in the one risky asset case.

7 References

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